SCALAR CURVATURE RIGIDITY OF ALMOST HERMITIAN SPIN MANIFOLDS WHICH ARE ASYMPTOTICALLY COMPLEX HYPERBOLIC

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ABSTRACT. This paper generalizes a rigidity result of complex hyperbolic spaces by M. Herzlich. We prove that an almost Hermitian spin manifold (M,g) of real dimension $4n+2$ which is strongly asymptotic to $\mathbb{C}\mathrm{H}^{2n+1}$ and satisfies a certain scalar curvature bound must be isometric to the complex hyperbolic space. The fact that we do not assume g to be Kähler reflects in the inequality for the scalar curvature.

1. Introduction

Rigidity of symmetric spaces of non–compact type is a frequently studied problem (cf. [2, 5, 9, 10]). Based on E. Witten's idea in the proof of the positive energy theorem (cf. [12]), R. Bartnik showed in [2] that an asymptotically flat spin manifold of non–negative scalar curvature and with vanishing mass must be the Euclidean space. The analogous rigidity result for the real hyperbolic was proved by M. Min–Oo in [9], in particular a strongly asymptotically hyperbolic spin manifold (M^n,g) with scalar curvature scal $\geq -n(n-1)$ is isometric to the hyperbolic space. Moreover, M. Herzlich showed in [5] that a strongly asymptotically complex hyperbolic Kähler spin manifold (M^{2m},g) of odd complex dimension m and with scalar curvature scal $\geq -4m(m+1)$ must be isometric to the complex hyperbolic space $\mathbb{C}\mathrm{H}^m$.

In this paper we generalize Herzlich's result in the way that we replace the Kähler assumption by the weaker condition: almost Hermitian.

Definition 1. $(\mathbb{C}\mathrm{H}^m, g_0)$ denotes the complex hyperbolic space of complex dimension m and holomorphic sectional curvature -4, i.e. $K \in [-4,-1]$, as well as $B_R(q) \subset M$ is the set of all $p \in M$ with geodesic distance to q less than R. Let (M^{2m}, g, J) be an almost Hermitian manifold, i.e. g is a Riemannian metric and J is a g–compatible almost complex structure. (M,g,J) is said to be *strongly asymptotically complex hyperbolic* if there is a compact manifold $C \subset M$ and a diffeomorphism $f : E := M - C \to \mathbb{C}\mathrm{H}^m - \overline{B_R(0)}$ in such a way that the positive definite gauge transformation

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$A \in \Gamma(\mathrm{End}(TM_{|E}))$ given by

$$g(AX, AY) = (f^*g_0)(X, Y) \quad g(AX, Y) = g(X, AY)$$

satisfies:

(1) A is uniformly bounded.
(2) Suppose r is the f^*g_0–distance to a fixed point, ∇^0 is the Levi–Civita connection for f^*g_0 and J_0 is the complex structure of $\mathbb{C}\mathrm{H}^m$ pulled back to E, then

$$\left|\nabla^0 A\right| + |A - \mathrm{Id}| + |AJ_0 - J| \in L^1(E; e^{2r}\mathrm{vol}_g) \cap L^2(E; e^{2r}\mathrm{vol}_g).$$

In particular, in contrast to the previous definition and result by Herzlich, a compact conformal transformation of the standard metric on $\mathbb{C}\mathrm{H}^m$ supplies a manifold which is strongly asymptotically complex hyperbolic.

Theorem 1. *Let (M^{4n+2}, g, J) be a complete almost Hermitian spin manifold of odd complex dimension $m = 2n+1$. If (M, g, J) is strongly asymptotically complex hyperbolic and satisfies the scalar curvature bound*

$$\mathrm{scal} \geq -4m(m+1) + 2\Big[|\mathrm{d}^*\Omega| + |\mathcal{D}'\Omega| + |\mathcal{D}''\Omega|\Big], \tag{1}$$

then (M, g, J) is Kähler and isometric to $\mathbb{C}\mathrm{H}^m$.

In this case $\Omega = g(.,J.)$ is the 2–form associated to J, d^* is formal L^2–adjoint of the exterior derivative d and $\mathcal{D}' + \mathcal{D}''$ is the Dolbeault decomposition of $\mathcal{D} = \mathrm{d} + \mathrm{d}^*$ in $\Lambda^*(TM) \otimes \mathbb{C}$, i.e. if $e_1, \dots, e_{2m}$ is an orthonormal base, we define $\mathcal{D}' = \sum e_j^{1,0} \cdot \nabla_{e_j}$ and $\mathcal{D}'' = \sum e_j^{0,1} \cdot \nabla_{e_j}$. Introduce $\mathcal{D}^c := \mathrm{d}^c + \mathrm{d}^{c,*}$ with $\mathrm{d}^c := \sum J(e_k) \wedge \nabla_{e_k}$ and $\mathrm{d}^{c,*} := -\sum J(e_k) \llcorner \nabla_{e_j}$, we obtain $\mathcal{D}' = \frac{1}{2}(\mathcal{D} - \mathbf{i}\mathcal{D}^c)$ as well as $\mathcal{D}'' = \frac{1}{2}(\mathcal{D} + \mathbf{i}\mathcal{D}^c)$. In particular, we can estimate

$$|\mathcal{D}'\Omega| + |\mathcal{D}''\Omega| \leq |\mathrm{d}^*\Omega| + |\mathrm{d}\Omega| + |\mathrm{d}^{c,*}\Omega| + |\mathrm{d}^c\Omega|.$$

The proof of this rigidity theorem is as usual based on the non–compact Bochner technique which was introduced by Witten in [12]. We show an integrated Bochner–Weitzenböck formula for the Kähler Killing connection which allows the usage of this technique. We expect to prove a similar result in the complex even–dimensional case and for the quaternionic hyperbolic space, but because of representation theoretical problems, there will be more terms involved in inequality (1).

2. Preliminaries

Let (M, g, J) be an almost Hermitian spin manifold of complex dimension m and denote by γ respectively $\cdot$ the Clifford multiplication on the complex spinor bundle $\$M$ of M. $\$M$ decomposes orthogonal into

$$\$M = \$_0 \oplus \cdots \oplus \$_m \tag{2}$$

(cf. [6, 8]) where each $\$_j$ is an eigenspace of $\Omega = g(.,J.)$ to the eigenvalue $\mathbf{i}(m-2j)$. We denote by π_j the orthogonal projection $\$M \to \$_j$. The

decomposition (2) is parallel (i.e. $\nabla\pi_j = 0$ for all j) if (g, J) is Kähler. As usual we introduce $X^{1,0} := \frac{1}{2}(X - \mathbf{i}J(X))$ as well as $X^{0,1} := \frac{1}{2}(X + \mathbf{i}J(X))$ and obtain $\gamma(X^{1,0}) : \$_j \to \$_{j+1}$ as well as $\gamma(X^{0,1}) : \$_j \to \$_{j-1}$, where $\$_j = \{0\}$ if $j \notin \{0, \dots, m\}$.

Supposing (g, J) to be Kähler and $m = 2n + 1$ to be odd, then a Kähler Killing spinor (cf. [6]) is a section in $\$_n \oplus \$_{n+1}$ which is parallel w.r.t.

$$\nabla_X + \kappa \left(\gamma(X^{1,0})\pi_n + \gamma(X^{0,1})\pi_{n+1}\right).$$

In particular, if there is a non–trivial Kähler Killing spinor, g is Einstein of scalar curvature $4m(m+1)\kappa^2$. Moreover, the subbundle $\$_n \oplus \$_{n+1}$ is trivialized by Kähler Killing spinors on $\mathbb{C}\mathrm{H}^m$ if we choose $\kappa = \pm\mathbf{i}$.

3. Bochner–Weitzenböck formula

Suppose (M, g, J) is spin and almost Hermitian of odd complex dimension $m = 2n+1$. We define $\mathcal{V} := \$_n \oplus \$_{n+1}$, its projection $\mathrm{pr}_{\mathcal{V}} := \pi_n + \pi_{n+1}$ and

$$\mathfrak{T}_X := \mathbf{i}\left(\gamma(X^{1,0})\pi_n + \gamma(X^{0,1})\pi_{n+1}\right).$$

Since $(\gamma(X^{1,0})\pi_j)^* = -\gamma(X^{0,1})\pi_{j+1}$, $\mathfrak{T}$ is a selfadjoint endomorphism on $\mathcal{V}$ (respectively $\$M$). Define the connection $\widehat{\nabla} := \nabla + \mathfrak{T}$ on $\$M$. The Dirac operator of $\widehat{\nabla}$ is given by $\widehat{\not{D}} = \not{D} + \mathcal{T}$ where $\not{D}$ is the Dirac operator of ∇ and $\mathcal{T}$ equals

$$-\mathbf{i}(m+1)\mathrm{pr}_{\mathcal{V}}$$

in this case we used (cf. [6])

$$\sum_k e_k \cdot e_k^{1,0} = -m + \mathbf{i}\gamma(\Omega) \quad \text{and} \quad \sum_k e_k \cdot e_k^{0,1} = -m - \mathbf{i}\gamma(\Omega). \tag{3}$$

Since $\gamma(X)\mathcal{T}$ is not selfadjoint on the full spinor bundle, we consider instead $\mathbb{T} := -\mathbf{i}(m+1)$ as well as the Dirac operator $\widetilde{\not{D}} := \not{D} + \mathbb{T}$.

Proposition 1. *Let (M, g, J) be almost Hermitian of odd complex dimension m, then the integrated Bochner–Weitzenböck formula*

$$\int_{\partial N} \left\langle \widehat{\nabla}_\nu \varphi + \nu \cdot \widetilde{\not{D}}\varphi, \psi \right\rangle = \int_N \left\langle \widehat{\nabla}\varphi, \widehat{\nabla}\psi \right\rangle - \left\langle \widetilde{\not{D}}\varphi, \widetilde{\not{D}}\psi \right\rangle + \left\langle \widehat{\mathfrak{R}}\varphi, \psi \right\rangle$$

holds for any compact $N \subset M$ and $\varphi, \psi \in \Gamma(\$M)$. In this case ν is the outward normal vector field on ∂N and $\widehat{\mathfrak{R}}$ is given by

$$\frac{\mathrm{scal}}{4} + m(m+1) + (m+1)^2 \mathrm{pr}_{\mathcal{V}^\perp} + \delta\mathfrak{T}$$

while $\mathrm{pr}_{\mathcal{V}^\perp}$ is the projection to the orthogonal complement of $\mathcal{V}$ in $\$M$ and $\delta\mathfrak{T}$ is the divergence of $\mathfrak{T}$, i.e. $\delta\mathfrak{T} = \sum(\nabla_{e_j}\mathfrak{T})_{e_j}$. Moreover, the boundary operator $\widehat{\nabla}_\nu + \nu \cdot \widetilde{\not{D}}$ is selfadjoint.

Proof. The essential facts are $(\mathfrak{T}_X)^* = \mathfrak{T}_X$ and $(\gamma(X)\mathbb{T})^* = \gamma(X)\mathbb{T}$. In particular, since $\nabla_\nu + \nu \cdot \not{D}$ is a selfadjoint boundary operator, $\widehat{\nabla}_\nu + \nu \cdot \widetilde{\not{D}}$ is selfadjoint. The formal L^2–adjoint of $\widetilde{\not{D}}$ is given by $\widetilde{\not{D}}^* = \not{D} - \mathbb{T}$. Thus, we can easily verify

$$\int_N \left\langle \widetilde{\not{D}}\varphi, \widetilde{\not{D}}\psi \right\rangle = -\int_{\partial N} \left\langle \nu \cdot \widetilde{\not{D}}\varphi, \psi \right\rangle + \int_N \left\langle \widetilde{\not{D}}^* \widetilde{\not{D}}\varphi, \psi \right\rangle$$

as well as $\widetilde{\not{D}}^*\widetilde{\not{D}} = \not{D}^2 + (m+1)^2$. Moreover, using $(\mathfrak{T}_X)^* = \mathfrak{T}_X$ on $\not{S}M$ leads to

$$\begin{aligned}
\int_N \left\langle \widehat{\nabla}\varphi, \widehat{\nabla}\psi \right\rangle &= \int_N \langle \nabla\varphi, \nabla\psi\rangle + \langle \nabla\varphi, \mathfrak{T}\psi\rangle + \langle \mathfrak{T}\varphi, \nabla\psi\rangle + \langle \mathfrak{T}\varphi, \mathfrak{T}\psi\rangle \\
&= \int_{\partial N} \langle \nabla_\nu\varphi + \mathfrak{T}_\nu\varphi, \psi\rangle + \int_N \langle \nabla^*\nabla\varphi, \psi\rangle + \\
&\quad + \int_N \langle \mathfrak{T}\varphi, \mathfrak{T}\psi\rangle - \langle \delta\mathfrak{T}\varphi, \psi\rangle
\end{aligned}$$

for all $\varphi, \psi \in \Gamma(\not{S}M)$. We use the facts $\pi_j\gamma(X)\pi_{j-1} = \gamma(X^{1,0})\pi_{j-1}$ and $\pi_j\gamma(X)\pi_{j+1} = \gamma(X^{0,1})\pi_{j+1}$ as well as (3) to compute

$$\begin{aligned}
\langle \mathfrak{T}\varphi, \mathfrak{T}\psi\rangle =& \sum_k \left\langle e_k \cdot \varphi_n, e_k^{1,0} \cdot \psi_n \right\rangle + \sum_k \left\langle e_k \cdot \varphi_{n+1}, e_k^{0,1} \cdot \psi_{n+1} \right\rangle \\
=&(m+1) \langle \mathrm{pr}_{\mathcal{V}}\varphi, \psi\rangle .
\end{aligned}$$

In particular, the Lichnerowicz formula $\not{D}^2 = \nabla^*\nabla + \frac{\mathrm{scal}}{4}$ gives the claim with

$$\widehat{\mathfrak{R}} = \frac{\mathrm{scal}}{4} + (m+1)^2 - (m+1)\mathrm{pr}_{\mathcal{V}} + \delta\mathfrak{T}.$$

□

Lemma 1. *Suppose inequality (1) of the main theorem holds, then at each point of M, $\widehat{\mathfrak{R}}$ has no negative eigenvalues: $\widehat{\mathfrak{R}} \geq 0$.*

Proof. We have to find an estimate for $\delta\mathfrak{T}$. Let $e_1, \ldots, e_{2m}$ be normal coordinates at T_pM with $e_{m+j} := Je_j$ in p. We obtain

$$\begin{aligned}
\delta\mathfrak{T} =& \sum_{j=1}^{2m} (\nabla_{e_j}\mathfrak{T})_{e_j} \\
=& \frac{1}{2}\gamma(\delta J)(\pi_n - \pi_{n+1}) + \mathbf{i}\sum_{j=1}^{2m} \left(e_j^{1,0} \cdot \nabla_{e_j}\pi_n + e_j^{0,1} \cdot \nabla_{e_j}\pi_{n+1} \right).
\end{aligned}$$

Thus, we have to estimate $\nabla_X\pi_r$ for $r = n, n+1$. We conclude from $\pi_n\gamma(\Omega) = \mathbf{i}\pi_n$

$$(\nabla_X\pi_n)(\mathbf{i} - \gamma(\Omega)) = \pi_n\gamma(\nabla_X\Omega)$$

as well as from $\gamma(\Omega)\pi_n = \mathbf{i}\pi_n$

$$(\mathbf{i} - \gamma(\Omega))(\nabla_X \pi_n) = \gamma(\nabla_X \Omega)\pi_n.$$

Using the facts $\pi_n(\nabla_X \pi_n)\pi_n = 0$ and $\mathbf{i} - \gamma(\Omega) = \sum_{j\neq n} c_j \pi_j$ with $|c_j| \geq 2$, $|\nabla_X \pi_n|$ can be estimated by $\frac{1}{2}|\nabla_X \Omega|$. Thus,

$$\sum_{j=1}^{2m} e_j^{1,0} \cdot (\nabla_{e_j}\pi_n)(\mathbf{i} - \gamma(\Omega)) = \pi_{n+1} \sum_{j=1}^{2m} \gamma(e_j^{1,0} \cdot \nabla_{e_j}\Omega)$$

leads to

$$\left| \sum_{j=1}^{2m} e_j^{1,0} (\nabla_{e_j}\pi_n)\phi \right| \leq \frac{1}{2} \left| \gamma(\mathcal{D}'\Omega)\phi \right|,$$

if $\pi_n(\phi) = 0$. Moreover,

$$\begin{aligned} \sum_{j=1}^{2m} \gamma(e_j^{1,0} \cdot \nabla_{e_j}\Omega)\pi_n &= \sum_{j=1}^{2m} \gamma(e_j^{1,0})(\mathbf{i} - \gamma(\Omega))(\nabla_{e_j}\pi_n) \\ &= -\sum_{j=1}^{2m} (\mathbf{i} + \gamma(\Omega))\gamma(e_j^{1,0})(\nabla_{e_j}\pi_n) \end{aligned}$$

shows

$$\left| \sum_{j=1}^{2m} e_j^{1,0} (\nabla_{e_j}\pi_n)\phi \right| \leq \frac{1}{2} \left| \gamma(\mathcal{D}'\Omega)\phi \right|,$$

if $\phi \in \mathbb{S}_n$, in this case we used $\pi_{n+1}(e_j^{1,0} \cdot \nabla_{e_j}\pi_n)\pi_n = 0$ and the fact that $\mathbf{i}+\gamma(\Omega)$ has absolute minimal eigenvalue 2 on $\mathbb{S}_{n+1}^{\perp}$ The same method applied to $\pi_{n+1}\gamma(\Omega) = -\mathbf{i}\pi_{n+1}$ and $\gamma(\Omega)\pi_{n+1} = -\mathbf{i}\pi_{n+1}$ yields

$$\left| \sum_{j=1}^{2m} e_j^{0,1} (\nabla_{e_j}\pi_{n+1}) \right| \leq \frac{1}{2} |\mathcal{D}''\Omega|.$$

Therefore, we obtain

$$|\delta\mathfrak{T}| \leq \frac{1}{2} \left(|\mathrm{d}^*\Omega| + |\mathcal{D}'\Omega| + |\mathcal{D}''\Omega| \right)$$

which gives the claim $\widehat{\mathfrak{R}} \geq 0$. □

4. Proof of the theorem

Lemma 2. *Suppose (M, g) is a complete spin manifold of real dimension $2m$. If the scalar curvature is uniformly bounded with* $\mathrm{scal} \geq -4m(m+1)$, *the Dirac operator*

$$\widetilde{\not{D}} = \not{D} - \mathbf{i}(m+1) : W^{1,2}(M, \mathbb{S}M) \to L^2(M, \mathbb{S}M)$$

is an isomorphism of Hilbert spaces.

Proof. (cf. [1, 5, 9]) Using the Lichnerowicz formula proves that the bilinear form $B(\varphi,\psi) = \int_M \left\langle \widetilde{\mathcal{D}}\varphi, \widetilde{\mathcal{D}}\psi \right\rangle$ is coercive and bounded on $W^{1,2}(M, \$M)$. The surjectivity of $\widetilde{\mathcal{D}}$ follows from the Riesz representation theorem and [4, Thm. 2.8]. □

Let (M, g, J) be an almost Hermitian spin manifold which is strongly asymptotically complex hyperbolic, where $E \subset M$ is supposed to be the Euclidean end of M. We consider the connection $\widehat{\nabla} = \nabla + \mathfrak{T}$ on $\$M$ and the connection $\widehat{\nabla}^0 = \nabla^0 + \mathfrak{T}^0$ on $\$M_{|E}$, where ∇^0 is the Levi–Civita connection and $\mathfrak{T}^0$ is the Kähler Killing structure for the complex hyperbolic metric on E. The bundle $\mathcal{V}^0 \subset \$M_{|E}$ is trivialized by spinors parallel w.r.t. $\widehat{\nabla}^0$.

The gauge transformation A extends to a bundle morphism $A : \$M_{|E} \to \$M_{|E}$ with (cf. [1])

$$\left|\overline{\nabla}\varphi - \nabla\varphi\right| \leq C \left|A^{-1}\right| \left|\nabla^0 A\right| |\varphi| \ ,$$

where ∇ is the usual spin connection for g and $\overline{\nabla}$ is a connection on $\$M_{|E}$ obtained from the connection $\overline{\nabla}$ on $TM_{|E}$ and given by $\overline{\nabla}Y = A(\nabla^0(A^{-1}Y))$.

Let ψ_0 be a spinor on $E \subset M$ which is parallel with respect to $\widehat{\nabla}^0$. Set $\psi := h(A\psi_0)$ for some cut off function h, i.e. $h = 1$ at infinity, $h = 0$ in $M - E$ and $\mathrm{supp}(\mathrm{d}h)$ compact. We compute

$$\begin{aligned}\widehat{\nabla}_X\psi =& (Xh)A\psi_0 + h(\nabla_X A\psi_0 + \mathfrak{T}_X(A\psi_0)) \\ =& (Xh)A\psi_0 + h(\nabla_X - \overline{\nabla}_X)A\psi_0 - hA\mathfrak{T}^0_X\psi_0 + h\mathfrak{T}_X A\psi_0\end{aligned}$$

and thus, the asymptotic assumptions supply

$$\widehat{\nabla}\psi \in L^2(M, T^*M \otimes \$M)$$

and

$$\left\langle \widehat{\nabla}_\nu\psi + \nu \cdot \widehat{\mathcal{D}}\psi, \psi \right\rangle \in L^1(M) \tag{4}$$

($|\psi_0|_0^2$ can be estimated by ce^{2r} for some $c > 0$). Using the above lemma gives a spinor $\xi \in W^{1,2}(M, \$M)$ with $\widetilde{\mathcal{D}}\xi = \widetilde{\mathcal{D}}\psi \in L^2$. In particular $\varphi := \psi - \xi$ is $\widetilde{\mathcal{D}}$–harmonic and non–trivial ($\psi \notin L^2$). Moreover, the selfadjointness of the boundary operator $\widehat{\nabla}_\nu + \nu \cdot \widetilde{\mathcal{D}}$ together with (4) implies as usual

$$\liminf_{r\to\infty} \int\limits_{\partial M_r} \left\langle \widehat{\nabla}_\nu\varphi + \nu \cdot \widetilde{\mathcal{D}}\varphi, \varphi \right\rangle = 0$$

for a non–degenerate exhaustion $\{M_r\}$ of M (cf. [1]). Since inequality (1) gives $\widehat{\mathfrak{R}} \geq 0$, we conclude from the integrated Bochner–Weitzenböck formula:

$$\int\limits_{\partial M_r} \left\langle \widehat{\nabla}_\nu\varphi + \nu \cdot \widetilde{\mathcal{D}}\varphi, \varphi \right\rangle \geq \int\limits_{M_r} \left|\widehat{\nabla}\varphi\right|^2 \geq 0,$$

that φ is parallel w.r.t. $\widehat{\nabla}$. Since $0 = \widehat{\not{\mathcal{D}}}\varphi = \not{\mathcal{D}}\varphi - \mathbf{i}(m+1)\mathrm{pr}_{\mathcal{V}}\varphi$ and $0 = \widetilde{\not{\mathcal{D}}}\varphi = \not{\mathcal{D}}\varphi - \mathbf{i}(m+1)\varphi$, we obtain that φ is a section of $\mathcal{V} = \$_n \oplus \$_{n+1}$. Furthermore, $\widehat{\nabla}^0$ is a flat connection of $\mathcal{V}^0$, so $\mathcal{V}$ is trivialized by spinors parallel w.r.t. $\widehat{\nabla}$. In particular, ∇_X preserves sections of $\mathcal{V}$. Since $\widehat{\nabla}$ is flat on $\mathcal{V}$, $\widehat{R} = 0$ implies

$$0 = R^s_{X,Y} + [\mathfrak{T}_X, \mathfrak{T}_Y] + (\nabla_X \mathfrak{T})_Y - (\nabla_Y \mathfrak{T})_X$$

on $\mathcal{V}$. A straightforward computations shows that $(\nabla_X \mathfrak{T})_Y$ and $(\nabla_Y \mathfrak{T})_X$ are Hermitian on $\mathcal{V}$ for all X, Y (use the fact $(\mathfrak{T}_X)^* = \mathfrak{T}_X$), but $R^s_{X,Y}$ as well as $[\mathfrak{T}_X, \mathfrak{T}_Y]$ are skew–Hermitian on $\mathcal{V}$ which leads to

$$0 = R^s_{X,Y} + [\mathfrak{T}_X, \mathfrak{T}_Y]. \tag{5}$$

From the fact (cf. [3])

$$\gamma(\mathrm{Ric}(X)) = 2\sum_i e_i \cdot R^s_{e_i,X}$$

and equation (5), we conclude $\mathrm{Ric}(X) = -2(m+1)X$ (cf. [6]), i.e. g is Einstein of scalar curvature $-4m(m+1)$. Inequality (1) yields $\mathrm{d}^*\Omega = 0$ as well as $\mathcal{D}'\Omega = 0$ and $\mathcal{D}''\Omega = 0$. In particular, $\mathcal{D}' + \mathcal{D}'' = \mathrm{d} + \mathrm{d}^*$ supplies $\mathrm{d}\Omega = 0$. Therefore, if J is integrable, (g, J) must be Kähler (cf. [7, p. 148]) and we could use the result by Herzlich to get the claim. However, we did not assume J to be integrable and in order to prove the general case, we compute the Riemannian curvature of (M, g, J). We have

$$\begin{aligned}[\mathfrak{T}_X, \mathfrak{T}_Y] =& (Y^{1,0} \cdot X^{0,1} \cdot - X^{1,0} \cdot Y^{0,1} \cdot)\pi_{n+1} + \\ &+ (Y^{0,1} \cdot X^{1,0} \cdot - X^{0,1} \cdot Y^{1,0} \cdot)\pi_n \\ =& -\frac{1}{2}\gamma(X \wedge Y + JX \wedge JY)(\pi_n + \pi_{n+1}) + \\ &+ \mathbf{i}\Omega(X,Y)\pi_{n+1} - \mathbf{i}\Omega(X,Y)\pi_n\end{aligned}$$

as well as $R^s_{X,Y} = \frac{1}{2}\gamma(\mathcal{R}(X \wedge Y))$, where $\mathcal{R}$ is the Riemannian curvature considered as endomorphism on $\Lambda^2 M$. Thus, we obtain

$$\gamma\big(\mathcal{R}(X \wedge Y) - X \wedge Y - JX \wedge JY - 2\Omega(X,Y)\Omega\big)\varphi = 0$$

for all $\varphi \in \Gamma(\mathcal{V})$ from (5) and $\gamma(\Omega)\pi_n = \mathbf{i}\pi_n$, $\gamma(\Omega)\pi_{n+1} = -\mathbf{i}\pi_{n+1}$. In particular, the following lemma shows

$$\mathrm{pr}_{\Lambda^{1,1}M} \circ \mathcal{R}(X \wedge Y) = X \wedge Y + JX \wedge JY + 2\Omega(X,Y)\Omega. \tag{6}$$

Lemma 3. *Suppose (V, q) is a vector space of real dimension $2m$ with a quadratic form q and a q–compatible complex structure J. Denote by $S = \oplus S_r$ the spinor space of V where S_r are induced from the action of the Kähler form Ω. Choose $l := \left[\frac{m-1}{2}\right]$, then if $\eta \in \Lambda^{1,1}V$ annihilates $S_l \oplus S_{l+1}$, i.e.*

$$\eta \cdot \psi = 0$$

for all $\psi \in S_l \oplus S_{l+1}$, η has to vanish.

Proof. Suppose that m is even. The only $\Lambda^{1,1}V$–forms which annihilate S_{l+1} are multiples of Ω (cf. [5]). But Ω acts as $2\mathbf{i}$ on S_l which shows the claim if m is even. Assume that m is odd and $\eta \cdot \psi = 0$ for all $\psi \in S_l \oplus S_{l+1}$. We consider the vector space $V \oplus \mathbb{C}^2$ with its spinor space $S\widehat{\otimes}\mathbb{C}^2$. Since Clifford multiplication with $\Lambda^2 V$ satisfies

$$\omega \cdot (\psi \otimes \varphi) = (\omega \cdot \psi) \otimes \varphi,$$

and $(S\widehat{\otimes}\mathbb{C}^2)_{l+1}$ is given by $S_l \otimes \mathbb{C} \oplus S_{l+1} \otimes \mathbb{C}$, we obtain $\eta \cdot \Psi = 0$ for all $\Psi \in (S\widehat{\otimes}\mathbb{C}^2)_{l+1}$. Thus, $\Lambda^{1,1}V \subset \Lambda^{1,1}(V \oplus \mathbb{C}^2)$ together with the above case (m even) implies that η is a multiple of Ω. But Ω acts as $\pm\mathbf{i}$ on S_l respectively S_{l+1} which shows the claim: $\eta = 0$. □

Using equation (6), the symmetry of the Riemannian curvature tensor and $\Omega \in \Gamma(\Lambda^{1,1}M)$ lead to

$$\langle \mathcal{R}(\Omega), X \wedge Y \rangle = \langle \mathcal{R}(X \wedge Y), \Omega \rangle = 2(m+1)\Omega(X,Y).$$

Consider the Bochner–Weitzenböck formula on $\Lambda^2 M$:

$$\triangle = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^* = \nabla^*\nabla + \mathfrak{R},$$

then $\mathfrak{R}$ is given by $\mathrm{Ric} + 2\mathcal{R}$ (cf. [11, Ap. B]), where Ric acts as derivation on $\Lambda^2 M$. We already know, that g is Einstein, i.e. $\mathrm{Ric} = -4(m+1)\mathrm{Id}_{\Lambda^2 M}$ supplies $\mathfrak{R}(\Omega) = 0$. Moreover, $\mathrm{d}\Omega = 0$ and $\mathrm{d}^*\Omega = 0$ imply that Ω is harmonic: $\triangle\Omega = 0$, i.e. we obtain $\nabla^*\nabla\Omega = 0$. Using the fact

$$0 = \triangle|\Omega|^2 = \mathrm{d}^*\mathrm{d}|\Omega|^2 = 2\langle \nabla^*\nabla\Omega, \Omega\rangle - 2\langle \nabla\Omega, \nabla\Omega\rangle$$

we conclude that (g, J) is Kähler. Thus, $\mathcal{R} : \Lambda^2 M \to \Lambda^{1,1}M$ together with (6) yield constant holomorphic sectional curvature -4 of (M, g, J). Since the end of M is diffeomorphic to $\mathbb{R}^{2m} - \overline{B_R(0)}$, M must be isometric to $\mathbb{C}\mathrm{H}^m$.

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